ON AN EXACT SOLUTION OF THE EQUATIONS OF MAGNETO-GAS-DYNAMICS OF FINITE CONDUCTIVITY

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Considered is a system of equations of magneto-gas-dynamics which describes plane unsteady flow in a magnetic field which is perpendicular to the plane of the flow. For the relation $\gamma = c_p/c_v = 2$, there is given a transformation which depends on one arbitrary time function and which reduces the system of equations of magneto-gas-dynamics to the same type of system but with a certain external force in the equation of motion.

To each solution of the new system of equations there corresponds a solution of the original system.

As an example, there is considered an exact solution which describes the compression of a plasma cylinder of finite conductivity.

1. The equations of magneto-gas-dynamics of a compressible liquid with finite conductivity in a transverse magnetic field $\mathbf{H}(0,0,H)$ have the form [1]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{H^2}{8\pi}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{H^2}{8\pi}\right)$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \nu_m \Delta H \qquad \left(\nu_m = \frac{c^2}{4\pi\sigma}\right) \qquad (1.1)$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + 2p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{\nu_m}{4\pi}\left[\left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2\right]$$

Here, t is time, p pressure, ρ density; x, y, are rectangular Cartesian coordinates, u and v components of the velocity vector; H is the potential of the magnetic field, σ the conductivity.

Let us introduce new variables and new functions with the aid of the relations

$$\xi = f(t)\,x, \quad \eta = f(t)\,y, \quad \tau = \int_{t_0}^{t} f^2(t)\,dt, \quad \rho_1 = \frac{\rho}{f^2}, \quad H_1 = \frac{H}{f^2}$$

$$p_1 = \frac{p}{f^4}, \qquad u_1 = \frac{u}{f} + \frac{f'}{f^2}x, \qquad v_1 = \frac{v}{f} + \frac{f'}{f^2}y \tag{1.2}$$

For the differential operations we have

$$\frac{\partial}{\partial x} = f\frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial y} = f\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = f^2\frac{\partial}{\partial \tau} + xf'\frac{\partial}{\partial \xi} + yf'\frac{\partial}{\partial \eta}$$
$$\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} = f^2\left(\frac{\partial}{\partial \tau} + u_1\frac{\partial}{\partial \xi} + v_1\frac{\partial}{\partial \eta}\right) \tag{1.3}$$

If we make use of these relations, the system (1.1) can be written in the following form in terms of the new functions and variables:

$$\frac{\partial u_1}{\partial \tau} + u_1\frac{\partial u_1}{\partial \xi} + v_1\frac{\partial u_1}{\partial \eta} = -\frac{1}{\rho_1}\frac{\partial}{\partial \xi}\left(p_1 + \frac{H_1^2}{8\pi}\right) + \frac{1}{f}\frac{d^2 f}{d\tau^2}\xi$$

$$\frac{\partial v_1}{\partial \tau} + u_1\frac{\partial v_1}{\partial \xi} + v_1\frac{\partial v_1}{\partial \eta} = -\frac{1}{\rho_1}\frac{\partial}{\partial \eta}\left(p_1 + \frac{H_1^2}{8\pi}\right) + \frac{1}{f}\frac{d^2 f}{d\tau^2}\eta$$

$$\frac{\partial \rho_1}{\partial \tau} + u_1\frac{\partial \rho_1}{\partial \xi} + v_1\frac{\partial \rho_1}{\partial \eta} + \rho_1\left(\frac{\partial u_1}{\partial \xi} + \frac{\partial v_1}{\partial \eta}\right) = 0$$

$$\frac{\partial H_1}{\partial \tau} + u_1\frac{\partial H_1}{\partial \xi} + v_1\frac{\partial H_1}{\partial \eta} + H_1\left(\frac{\partial u_1}{\partial \xi} + \frac{\partial v_1}{\partial \eta}\right) = \nu_m \Delta H_1 \tag{1.4}$$

$$\frac{\partial p_1}{\partial \tau} + u_1\frac{\partial p_1}{\partial \xi} + v_1\frac{\partial p_1}{\partial \eta} + 2p_1\left(\frac{\partial u_1}{\partial \xi} + \frac{\partial v_1}{\partial \eta}\right) = \frac{\nu_m}{4\pi}\left[\left(\frac{\partial H_1}{\partial \xi}\right)^2 + \left(\frac{\partial H_1}{\partial \eta}\right)^2\right]$$

The system of equations of magneto-gas-dynamics (1.1) has been transformed into a similar system but with a spatial external force in the equations of motion.

The system (1.1) describes the motion of a conducting gas in regions where the motion is continuous. If the region of motion contains a surface of discontinuity, then the solutions of (1.1) on the two sides of this surface must be associated by means of definite relations. Under finite conductivity conditions the potential of the magnetic field must be continuous everywhere, $[\mathbf{H}] = 0$; ; the tangential component of the potential of the electric field $[\mathbf{E}_\tau] = 0$ must be continuous at the passage through the surface of discontinuity (the square brackets indicate that one takes the difference on both sides of the surface of discontinuity).

Let $F(t,x,y) = 0$ be the equation of the surface of a strong discontinuity. On this surface the following conditions of dynamic compatibility must be satisfied:

$$\left[\rho\left(\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y}\right)\right] = 0$$

$$\rho\left(\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y}\right)[\mathbf{V}] = -[p]\left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j}\right) \tag{1.5}$$

$$\rho\left(\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y}\right)\left[\frac{\mathbf{V}\cdot\mathbf{V}}{2} + \frac{p}{\rho}\right] = -\left[p\left(u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y}\right)\right]$$

The condition $[\mathbf{E}_\tau] = 0$ can be written as

$$\left[H\left(\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y}\right)\right] = \left[\nu_m\left(\frac{\partial H}{\partial y}\frac{\partial F}{\partial y} + \frac{\partial H}{\partial x}\frac{\partial F}{\partial x}\right)\right] \tag{1.6}$$

Under the transformation (1.2) the surface of the strong discontinuity $F(t,x,y) = 0$ becomes $F_1(\tau,\xi,\eta) = 0$ in terms of the new variables τ,ξ,η, where F_1 is such that

$$F_1\left(\int_{t_0}^{t} f^2\,dt,\ fx,\ fy\right) = F(t,x,y)$$

Making use of the relations (1.2) and (1.3) we can easily show that conditions (1.5) and (1.6) remain invariant under the transformation (1.2). One can obtain this result by rewriting (1.5) and (1.6) replacing t,x,y by τ,ξ,η and ρ,u,v,V,p,H,F by $\rho_1,u_1,v_1,V_1,p_1,H_1,F_1$.

The conditions of impenetrability through the solid wall, and the condition of the contacting discontinuity remain also invariant.

From what has been said, it follows that to each solution of the system (1.4) which describes a certain motion in the field of action of external spatial forces there corresponds a certain solution of the system of equations (1.1).

If we determine f from $d^2f/d\tau^2 = 0$, we obtain the invariance of not only the boundary conditions but also the invariance of the system of equations of magneto-gas-dynamics. Invariant transformations of the equations of gas-dynamics were considered in [2 and 3]

It should also be mentioned that in the flow region there may be present only dielectric bodies, or bodies which have infinite conductivity. For finite conductivity, the magnetic field inside the body is described by an equation which is not invariant under the transformation (1.2). Hence, such bodies are excluded from consideration.

2. As one of the simplest examples which gives a solution of the system of equations (1.4), we shall consider the equilibrium of a conducting circular cylinder bounded by a dielectric wall and located in a magnetic field. The magnetic force lines are parallel to the axis of the cylinder. Since the equilibrium is axially symmetric, we can reduce the system of equations (1.4) to polar coordinates, and we may assume that all quantities are independent of the angle.

Setting the velocity components equal to zero, we obtain the system of equilibrium equations (*)

$$\frac{\partial}{\partial\zeta}\left(p_1 + \frac{H_1^2}{8\pi}\right) = \frac{1}{f}\frac{\partial^2 f}{\partial\tau^2}\rho_1\zeta, \qquad \frac{\partial\rho_1}{\partial\tau} = 0$$

$$\frac{\partial H_1}{\partial\tau} = \frac{\nu_m}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial H_1}{\partial\zeta}\right), \qquad \frac{\partial p_1}{\partial\tau} = \frac{\nu_m}{4\pi}\left(\frac{\partial H_1}{\partial\zeta}\right)^2 \qquad (\zeta^2 = \xi^2 + \eta^2) \tag{2.1}$$

The system of equations (2.1) is, in general, over-determined, but one may, nevertheless, find particular solutions which will satisfy all the equations. We seek the solution for H_1 in the form

$$H_1 = e^{a\tau}h(\zeta).$$

Substituting this into the third equation of the system (2.1) we obtain

$$\frac{d^2h}{d\zeta^2} + \frac{1}{\zeta}\frac{dh}{d\zeta} - \varkappa^2 h = 0 \qquad \left(\varkappa^2 = \frac{a}{\nu_m},\ a > 0\right) \tag{2.2}$$

This equation has a solution which is bounded at the origin. It is the Bessel function $I_0(\varkappa\zeta)$ of imaginary argument. Thus,

$$H_1 = c_1 I_0(\varkappa\zeta)\,e^{a\tau} \tag{2.3}$$

where c_1 is an arbitray constant.

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*) Equilibrium will occur for the introduced fictitious medium with the parameters u_1, H_1, p_1, ..., while this medium, considered as a real medium, is in motion, which will be determined later.

The quantities p_1, and ρ_1 are easily determined from the remaining equations, and they have the form

$$p_1 = p_0 + \frac{e^{2a\tau}}{8\pi} c_1^2 I_1^2(\varkappa\zeta), \qquad \rho_1 = \frac{2c_1^2\varkappa}{B\zeta} I_1(\varkappa\zeta)\left[\frac{3}{2} I_0(\varkappa\zeta) + \frac{1}{2} I_2(\varkappa\zeta)\right] \quad (2.4)$$

where B is an arbitrary positive constant. Hereby $f(\tau)$ must satisfy Equation

$$d^2f / d\tau^2 - Be^{2a\tau} f = 0$$

We take one of the solutions of this equation in the form

$$f(\tau) = c_3 I_0(\beta e^{a\tau}) \qquad (\beta = \sqrt{B} / a) \qquad (2.5)$$

The obtained solution describes the following "equilibrium" state of the fictitious medium. Inside the cylinder of some radius ζ_0 there is a quiescent conducting gas whose density is distributed according to the law (2.4) The conducting gas is bounded by solid dielectric walls. At the instant of time $\tau = -\infty$ the pressure in the entire cylinder is constant and equal to p_0, the potential of the magnetic field is zero and the spatial external force vanishes, then there appears a magnetic field in the region $\zeta \geqslant \zeta_0$. This field increases according to the law $H_1 = c_1 I_0(\varkappa\zeta_0) e^{a\tau}$ Inside the region $\zeta \leqslant \zeta_0$ the magnetic field is determined by Formula (2.3). Simultaneously with the magnetic field, the external force is beginning to act. This force counterbalances the gradients of the magnetic and hydrodynamic pressure. The hydrodynamic pressure p_1, which is determined by Formula (2.4) increases continuously because of the heating of the gas by Joule's dissipation.

Let us now see what solution of the system of equations (1.1) will correspond to the solution of (1.4).

In order to pass from the derived solution to the solution of the system of equations (1.1), we use the relations (1.2) which for the case under consideration takes on the form

$$\zeta = fr, \quad \tau = \int_{t_0}^{t} f^2\,dt, \quad \rho = f^2\rho_1, \quad H = f^2 H_1, \quad p = f^4 p_1$$

$$w = -\frac{f'}{f} r \qquad (w^2 = u^2 + v^2)$$

The expression for f in terms of τ is given by Formula (2.5). The solution can be written in the form

$$w = -a\beta\, c_3^2 e^{a\tau} I_0(\beta e^{a\tau}) I_1(\beta e^{a\tau})\, r, \qquad H = c_1 c_3^2 e^{a\tau} I_0^2(\beta e^{a\tau}) I_0[\varkappa c_3 I_0(\beta e^{a\tau})\, r]$$

$$p = c_3^4 I_0^4(\beta e^{a\tau}) \left\{p_0 + \frac{c_1^2}{8\pi} e^{2a\tau} I_1^2[\varkappa c_3 I_0(\beta e^{a\tau})\, r]\right\}$$

$$\rho = \frac{c_3 c_1^2 \varkappa}{a^2\beta^2 r} I_0(\beta e^{a\tau}) I_1[\varkappa c_3 I_0(\beta e^{a\tau})\, r]\, \{3 I_0[\varkappa c_3 r I_0(\beta e^{a\tau})] + I_2[\varkappa c_3 I_0(\beta e^{a\tau})\, r]\} \quad (2.6)$$

The relation betweem the variables τ and t is given by $d\tau/dt = f^2$ Integrating this equation we get

$$t = \frac{1}{ac_3^2}\left[\frac{K_0(\beta)}{I_0(\beta)} - \frac{K_0(\beta e^{a\tau})}{I_0(\beta e^{a\tau})}\right] \qquad (\beta = \sqrt{B} / a) \qquad (2.7)$$

For the sake of definiteness, the constant is chosen so that to the time $t = 0$ there corresponds $\tau = 0$. To the value of the parameter $\tau = -\infty$ there corresonds the instant of time $t = -\infty$. To the value $\tau = +\infty$ there corresponds the instant of time

$$t = \frac{1}{ac_3^2}\frac{K_0(\beta)}{I_0(\beta)}$$

The solution of the system of equations (1.1), which is given by Formulas (2.6) and (2.7), describes the following magneto-gas-dynamic motion.

At the instant $t = -\infty$ the conducting gas is at rest in a circular cylinder of radius $r_0 = a_3\zeta_0$ The density of the gas is distributed according to a law which can be obtained if in the last expression of (2.6) one lets $\tau = -\infty$ The pressure is constant in the entire cross section and the magnetic field is absent, then there appears in the region $r \geqslant r_0$ a magnetic field which increases according to the law defined by the second expression of (2.6) where one has to set $r = f\zeta_0$. The magnetic field penetrates into the conducting gas and sets it in motion.

The distribution of velocities is given by the first formula of (2.6) which describes compression of the plasma cylinder. The law of motion of the dielectric wall is determined by Equation

$$r_c = c_3 I_0 (\beta e^{a\tau})\, \zeta_0 \qquad (c_3 > 0)$$

At the instant of time

$$t = \frac{1}{ac_3^2}\,\frac{K_0(\beta)}{I_0(\beta)}$$

the entire gas converges to the origin of the coordinate system.

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